n-Tokyoites' Loop-Line Commuter Problem

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Abstract

We present a new $O(n^2)$ order algorithm to an n-Tokyoites' Loop-Line Commuter Problem. The n-Tokyoites' Loop-Line Commuter Problem comprises a special class of the more general Gilmore-Gomory weighted bipartite matching problem where weights assigned to arcs are given in terms of integrals of some functions. The algorithm of $O(n^2)$ complexity developed is faster than the more popularly used Hungarian-type $O(n^3)$ algorithms [1,2] applicable to the more general weighted bipartite matching problem, but is slower than the original, more restricted Gilmore-Gomory $O(n \log n)$ algorithm [3]. The algorithm we have developed allows to impose some novel angular constraints which find an immediate application not only to the n-Tokyoites' Loop-Line Commuter Problem itself, but also to the data association problem involved in the multisensor-multitarget tracking process [4] and to the specifically defined Gilmore-Gomory's original TSP problem.

Key words: algorithm designing, weighted bipartite matching, Gilmore-Gomory matching problem, Hungarian algorithm, data assignment problem

1 Introduction and Problem Statement

The weighted bipartite matching problem can be described as follows: Given a bipartite graph $G = (S, T, S \times T)$, with each arc in $S \times T$ being associated with weights given by real numbers, it is required to find a complete matching (that is to say, each node in S and T to be incident to an arc of the match exactly once) for which the sum of the weights assigned to each of the arcs is a minimum (or a maximum). Depending on the weight function specified, a variety of applications emerge.

The weighted bipartite matching problem has long been a subject of extensive investigations in both operations research and classical combinatorial analysis

([5,6]), from widely differing motivations in various fields. The data assignment problem ([1,2]) which is one of the earliest optimization problems studied in the field of operations research is one such an example. And the general weighted bipartite matching problem emerges in various applications including the transportation problem ([7]), the traveling salesman problem ([3]), the data association problem involved in the multisensor-multitarget tracking process ([4]), and the scheduling problems among others which constitute the well known NP-hard problems. The Hungarian type algorithms have been used most extensively for the data assignment problem but most of the efforts made have been expended on improving the classical algorithm [1,8] of $O(n^3)$, resulting in improved algorithms such as the improved Hungarian algorithm [9], Auction and RELAX-II algorithms ([10,11]), and the signature methods [12,13] all of the $O(n^3)$ complexity [1] with an improvement in coefficients only. Vaidya[14] develops an $O(n^{2.5} \log n)$ time algorithm if the nodes lie in the plane and the weight function is defined as an Euclidean distance.

As noted by Gilmore and Gomory [3], Karp and Li [15], Aggarwal et al [7], Buss and Yianilos [16] and others, the analysis becomes substantially easier when the nodes lie on a line or on any curve homeomorphic to a circle. Karp and Li [15], and Aggarwal et al [7] have studied the case when the weights of arcs between the two nodes are set equal to the length of the shortest arc between them. This special matching problem is known as 'Skis and Skiers' problem of Lawler [5], where the sum of the absolute differences between the heights of each skier and his/her skis is to be minimized. Presorting the nodes, Karp and Li [15] have given a linear time algorithm to this matching problem. Aggarwal et al have generalized the linear time algorithm to the transportation problem, obtaining an $O(n \log n)$ algorithm. Buss and Yianilos [16] extend the restriction from "circle" to the "quasi-convex tour" graph and obtain an algorithm of $O(n \log n)$ complexity.

By restricting the nodes to lie on a line, Gilmore and Gomory [3] have studied a slightly more general class of the matching problem by extending the weight function assigned to the arc between two nodes to an integral of some functions f and g, depending on the dominating relation prevailing between the two nodes. Their algorithm has the complexity of $O(n \log n)$ for $n = |S| = |T|$, where most of the computing time is spent on sorting. Here again, we do not include the time complexity needed in computing w_{ij}.

This paper generalizes and extends the specialized Gilmore-Gomory matching problem to a more general class, called n-Tokyoites' Loop-Line Commuter Problem, allowing the nodes lying along the circle or actually any curve homeomorphic to a circle to accommodate novel angular constraints.

[1] To be strict, it is $O(|S|^2|T|)$, if $|S| \neq |T|$.

The n-Tokyoites' Loop-Line Commuter Problem is defined below[2].

n-Tokyoites' Loop-Line Commuter Problem: Suppose that two big Loop Railway lines are the only means of commuting in a big city where n skilful workers living along the railway line must commute to n-skill stations also along the circle. One line runs clockwise while the other runs counterclockwise with the two lines being managed by two independent managements so that the traveling fees between two stations need not be the same between clockwise and counterclockwise lines. Workers like to commute a shortest way and the company which pays for the traveling cost of skilful employees like to minimize the total traveling cost. Now we need matching these n men to the n work stations in such a way that the workers can always choose the shortest way, while the company will spend the minimum sum of money for the travel costs.

The n-Tokyoites' Loop-Line Commuter Problem discussed above can be formalized as follows: Let $G = (S, T, S \times T)$ be a complete bipartite graph with $|S| = |T| = n$. Each node $i \in S$ has associated with it a point α_i on a unit circle centered at O and each node $j \in T$ a point β_j on the unit circle. For all i, j, the weight of the arc (i, j) is taken to be

$$w_{ij} = \begin{cases} \int_{\alpha_i}^{\beta_j} f(s)ds, \text{ if } 0 \leq \angle\alpha_i\beta_j < \pi; \\ \int_{\beta_j}^{\alpha_i} g(s)ds, \text{ otherwise.} \end{cases} \tag{1}$$

where an angle $\angle ab$ is defined as the angle that the vector $\overrightarrow{Oa}$ must traverse to reach $\overrightarrow{Ob}$ in the clockwise direction and the integral $\int_a^b \cdot ds$ is defined along the path of the circle from a to b in the clockwise direction. Here $f(s)$, and $g(s)$ are any integrable functions satisfying $f(s) + g(s) \geq 0$ for all s [3]. It is required to find a minimum cost matching, that is, a subset X of $S \times T$ such that for each node in $S \cup T$, there is one and only one arc incident to it, for which the sum of the weights of the arcs is a minimum.

We give an $O(n^2)$ algorithm for the n-Tokyoites' Loop-Line Commuter Problem as compared with the $O(n^3)$ conventional Hungarian-type algorithm.

The n-Tokyoites' Loop-Line Commuter Problem we have formulated above has an immediate extension to the bearing data association problem in a multisensor-multitarget tracking environment from bearings-only measurements

[2] We call it an n-Tokyoites' Loop-Line Commuter problem because Tokyoites were used to take one Loop-Line Railway to work in the downtown area of Tokyo.

[3] It is possible that $f(s)$ ($g(s)$) be smaller than 0 at certain points; in some particular areas as in Las Vegas, gambling companies/hotel owners may be willing to pay back railway fees to commuters in order to promote some activities such as gambling with some risk of loss of course.

where an optimal solution is sought by a maximum likelihood method. In the bearing data association problem, a set of real bearing measurements are given, at any observing time, as $\{\beta_1, \beta_2, \cdots, \beta_n\}$ which are in turn obtained from front-end direction-of-arrival (DOA) estimators of the sensors and a set of estimated bearing measurements $\{\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_n\}$, giving another set of directions which are calculated from an estimation of the positions of the targets. We seek an optimal solution such that the difference between the estimated targets' positions and the real bearing measurements be a minimum. To do this, we firstly should find a match between the two sets $i \to c(i)$ ($c(i) \in \{1, 2, \cdots, n\}$, $c(i) \neq c(j)$ when $i \neq j$), such that

$$E = \sum_{i=1}^{j} (\angle \hat{\beta}_{c(i)} \beta_i)^2$$

is minimized.

To widen the class of the weight functions as expressed by equation (1), we have added the class of functions as expressed by equation (9) in the corollary of section II so that our analysis now includes the weight function applicable to the data association of the multisensor-multitarget tracking process. All the theorems and algorithms obtained in the paper remain valid for the tracking problem.

In the original Gilmore-Gomory matching problem, the weight w_{ij} of the arc $i \in S$ and $j \in T$ is associated with real numbers as defined by one of the integrals in formula (1) depending on whether $\alpha_i \leq \beta_j$. Reducing the real numbers by an appropriate factor such that we always have $-\pi < \alpha_i - \beta_j < \pi$ for all $i \in S$ and $j \in T$, it is easy to show that the Gilmore-Gomory matching problem is a special case of the problem described here.

Our work differs from the existing works of Karp and Li, Aggarwal et al, and Buss and Yianilos in one important point: While a nested matching ([7]) is allowed as an optimal matching in their works based on their definition of the weight as an arc length or a constraint of 'quasi-convex tour', this is not always true but can be true only occasionally in our cases. The difference comes from the class of admissible weight functions which the algorithms can accommodate. The difference also explains why a complicated enumeration is needed in our proofs of the theorems and lemmas because the proof by a simple and straightforward removal of all the non-nested matches or all the nested matches can not be used. It is easy to see that the n-Tokyoites' Loop-Line Commuter Problem can be regarded as the general case of the the matching problem studied by Karp and Li, and Aggarwal et al which corresponds to the situation that $f = g = 1$ in equation (1) [4]. The n-Tokyoites' Loop-Line

[4] Although it is possible to find other possible extensions besides $f = g = 1$, their

Commuter Problem as extended by us is much more flexible and is capable of widening the field of applications by accommodating a much wider class of weight functions including that of equation (9) as defined in the corollary of Section II. In sharp contrast, the algorithms of Karp and Li, Aggarwal et al, Buss and Yianilos remain valid only for the extreme case of $\delta = 0$ of our class.

2 Main Result

Definition 1 (Rotational Sort) *Consider a list of distinct points on the circle $a_1, a_2, \cdots, a_n$. We define a rotationally sorted list of these points as a permutation $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$ such that $\angle a_{i_1} a_{i_j} \leq \angle a_{i_1} a_{i_k}$, for any $1 \leq j < k \leq n$.*

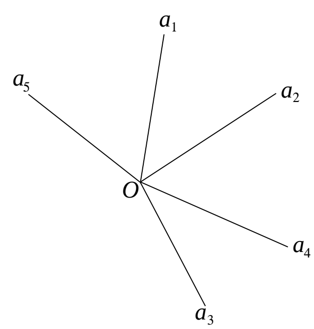

Fig. 1. An Example of Rotational Sort

It should be important to note that even if $a_1, a_2, \cdots, a_n$ are all distinct, the rotationally sorted list is not unique. In fact, there are n rotationally sorted lists in this case. For example, while a_3, a_5, a_1, a_2, a_4 is a rotationally sorted list of the points a_1, a_2, a_3, a_4, a_5 in figure 1, a_2, a_4, a_3, a_5, a_1 is also a rotationally sorted list of the same points.

The notation $a_1 \prec a_2 \prec \cdots \prec a_n$ implies that a list of $a_1, a_2, \cdots, a_n$ has been rotationally sorted. For a given rotationally sorted list of points, we can always use "$\prec$" to denote a sequence of any two points in the list.

To simplify the notation in the subsequent lemmas and theorems to follow, we write for any complete matching X,

$$w(X) = \sum_{(i,j) \in X} w_{ij}. \tag{2}$$

Lemma 1 *Suppose that $\alpha_{i_1} \neq \alpha_{i_2}$ and $\beta_{j_1} \neq \beta_{j_2}$, and α_{i_1}, α_{i_2}, β_{j_1}, and β_{j_2}, satisfy property (1), property (2) or (2'), and property (3) or (3') below:*

(1) There is a rotational sort of α_{i_1}, α_{i_2}, β_{j_1}, and β_{j_2}, such that $\beta_{j_1} \prec \beta_{j_2}$ and $\alpha_{i_1} \prec \alpha_{i_2}$;

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algorithms are still far more restricted than our algorithm

(2) $0 \le \angle\alpha_{i_1}\beta_{j_1} < \pi$, $0 \le \angle\alpha_{i_1}\beta_{j_2} < \pi$, *and* $0 \le \angle\beta_{j_1}\beta_{j_2} < \pi$;
(2') $\pi \le \angle\alpha_{i_1}\beta_{j_1} < 2\pi$, $\pi \le \angle\alpha_{i_2}\beta_{j_1} < 2\pi$, *and* $0 \le \angle\alpha_{i_1}\alpha_{i_2} < \pi$;
(3) $\pi \le \angle\alpha_{i_2}\beta_{j_2} < 2\pi$, $\pi \le \angle\alpha_{i_2}\beta_{j_1} < 2\pi$, *and* $0 \le \angle\beta_{j_1}\beta_{j_2} < \pi$;
(3') $0 \le \angle\alpha_{i_2}\beta_{j_2} < \pi$, $0 \le \angle\alpha_{i_1}\beta_{j_2} < \pi$, *and* $0 \le \angle\alpha_{i_1}\alpha_{i_2} < \pi$.

Then

$$w_{i_1,j_1} + w_{i_2,j_2} \le w_{i_1,j_2} + w_{i_2,j_1}. \tag{3}$$

PROOF. Without loss of generality, we suppose that the property (2) is applicable. Then if case (3) is applicable, we have $\alpha_{i_1} \prec \beta_{j_1} \prec \beta_{j_2} \prec \alpha_{i_2}$ (see figure 2(1)) and

$$\begin{aligned}
&(w_{i_1,j_1} + w_{i_2,j_2}) - (w_{i_1,j_2} + w_{i_2,j_1}) \\
&= \int_{\alpha_{i_1}}^{\beta_{j_1}} f(s)ds + \int_{\beta_{j_2}}^{\alpha_{i_2}} g(s)ds - \int_{\alpha_{i_1}}^{\beta_{j_2}} f(s)ds - \int_{\beta_{j_1}}^{\alpha_{i_2}} g(s)ds \\
&= -\int_{\beta_{j_1}}^{\beta_{j_2}} (f(s) + g(s))ds \\
&\le 0.
\end{aligned} \tag{4}$$

If case (3') is applicable, we need to check two subcases of (a) $\alpha_{i_1} \prec \beta_{j_1} \prec \alpha_{i_2} \prec \beta_{j_2}$ (see figure 2(2)) and (b) $\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1} \prec \beta_{j_2}$ (see figure 2(3)). For subcase (a) , we have

$$\begin{aligned}
&(w_{i_1,j_1} + w_{i_2,j_2}) - (w_{i_1,j_2} + w_{i_2,j_1}) \\
&= \int_{\alpha_{i_1}}^{\beta_{j_1}} f(s)ds + \int_{\alpha_{i_2}}^{\beta_{j_2}} f(s)ds - \int_{\alpha_{i_1}}^{\beta_{j_2}} f(s)ds - \int_{\beta_{j_1}}^{\alpha_{i_2}} g(s)ds \\
&= -\int_{\beta_{j_1}}^{\alpha_{i_2}} (f(s) + g(s))ds \\
&\le 0.
\end{aligned} \tag{5}$$

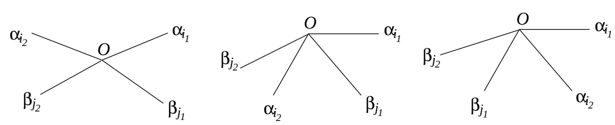
For subcase (b), we have

$$(w_{i_1,j_1} + w_{i_2,j_2}) - (w_{i_1,j_2} + w_{i_2,j_1})$$

$$
\begin{aligned}
&= \int_{\alpha_{i_1}}^{\beta_{j_1}} f(s)ds + \int_{\alpha_{i_2}}^{\beta_{j_2}} f(s)ds - \int_{\alpha_{i_1}}^{\beta_{j_2}} f(s)ds - \int_{\alpha_{i_2}}^{\beta_{j_1}} f(s)ds \\
&= 0. \qquad (6)
\end{aligned}
$$

The conclusion of the lemma is now immediate.

The inequality (3) is more popularly called a Monge property [17]. In fact we see easily that such Monge property holds for all the β_{j_1}, β_{j_2}, α_{i_1} and α_{i_2} in the original Gilmore-Gomory matching problem as long as the relation $\beta_{j_1} < \beta_{j_2}$ and $\alpha_{i_1} < \alpha_{i_2}$ holds. Lemma 1 implies that the Monge property holds under certain conditions in the n-Tokyoites' Loop-Line Commuter Problem . Under only these conditions where the Monge property holds, a matching X with $(i_1, j_2), (i_2, j_1) \in X$ could be replaced by a matching $X' = X - \{(i_1, j_2), (i_2, j_1)\} + \{(i_1, j_1), (i_2, j_2)\}$ without increasing the matching cost.



(1)$\alpha_{i_1} \prec \beta_{j_1} \prec \beta_{j_2} \prec \alpha_{i_2}$ (2)$\alpha_{i_1} \prec \beta_{j_1} \prec \alpha_{i_2} \prec \beta_{j_2}$ (3)$\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1} \prec \beta_{j_2}$

Fig. 2. The Cases of Lemma 1

Definition 2 *Suppose α_{i_1}, α_{i_2} and α_{i_3} as well as β_{j_1}, β_{j_2} and β_{j_3} are pairwise unequal elements in S and T, satisfying $\alpha_{i_1} \prec \alpha_{i_2} \prec \alpha_{i_3}$, and $\beta_{j_1} \prec \beta_{j_2} \prec \beta_{j_3}$, respectively; if $(i_1, j_1), (i_2, j_3), (i_3, j_2) \in X$, then we say that the matching X includes a crossover matching of i_1, i_2, i_3 and j_1, j_2, j_3.*

We should note that $\alpha_{i_1} \prec \alpha_{i_2} \prec \alpha_{i_3}$ is equivalent to $\alpha_{i_3} \prec \alpha_{i_1} \prec \alpha_{i_2}$ and $\alpha_{i_2} \prec \alpha_{i_3} \prec \alpha_{i_1}$, $\beta_{j_1} \prec \beta_{j_2} \prec \beta_{j_3}$ is equivalent to $\beta_{j_3} \prec \beta_{j_1} \prec \beta_{j_2}$ and $\beta_{j_2} \prec \beta_{j_3} \prec \beta_{j_1}$. So, the crossover does not depend on the rotationally sorted list selected.

Theorem 1 *Suppose the matching X includes a crossover matching. Then we could find a matching X' which does not include the crossover matching, such that $w(X') \leq w(X)$.*

PROOF. Suppose X includes the crossover matching of i_1, i_2, i_3 and j_1, j_2, j_3 as $(i_1, j_1), (i_2, j_3), (i_3, j_2) \in X$.

Without loss of generality, we can take $0 \leq \angle\alpha_{i_1}\beta_{j_1} < \pi$ (see figure 3).

For any point a on the unit circle, we denote $\bar{a}$ as the point on the circle such that $\angle a\bar{a} = \pi$.

To determine a possible position for α_{i_2}, it is sufficient to enumerate the following 4 cases of figures 3(1), 3(2), 3(3) and 3(4), representing $\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1}$, $\beta_{j_1} \prec \alpha_{i_2} \prec \bar{\alpha}_{i_1}$, $\bar{\alpha}_{i_1} \prec \alpha_{i_2} \prec \bar{\beta}_{j_1}$ and $\bar{\beta}_{j_1} \prec \alpha_{i_2} \prec \alpha_{i_1}$ separately.

For the first case of $\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1}$, we further consider the subcases depending on the positions of β_{j_3} (see figure 4). It is easy to prove Theorem 1 from Lemma 1 for this case. X can be replaced by $X' = X - \{(i_1, j_1), (i_2, j_3)\} + \{(i_1, j_3), (i_2, j_1)\}$ which has no crossover matching of i_1, i_2, i_3 and j_1, j_2, j_3 for the cases of figures 4(1) and (2), because of $(i_1, j_1), (i_2, j_3) \in X$; the case of figure 4(3) must be subdivided further into the subcases of 4(3)-a, 4(3)-b and 4(3)-c, where $(i_2, j_3), (i_3, j_2) \in X$, $(i_2, j_3), (i_3, j_2) \in X$ and $(i_1, j_1), (i_3, j_2) \in X$ suggest the replacements of X' separately.

For the cases of $\beta_{j_1} \prec \alpha_{i_2} \prec \bar{\alpha}_{i_1}$, $\bar{\alpha}_{i_1} \prec \alpha_{i_2} \prec \bar{\beta}_{j_1}$ and $\bar{\beta}_{j_1} \prec \alpha_{i_2} \prec \alpha_{i_1}$ of figures 5, 6 and 7 respectively, we could always find a piece of evidence in each case that X can now be replaced by X' in accordance with Lemma 1.

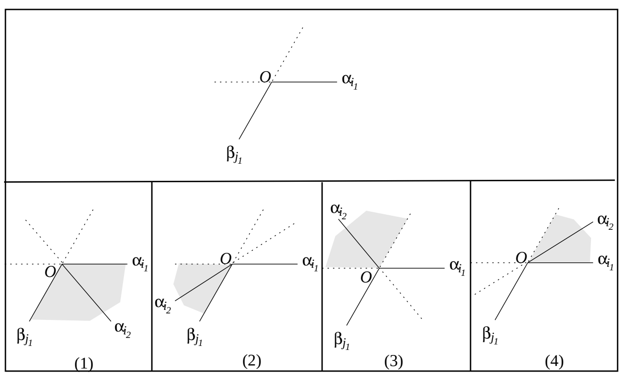


Fig. 3. The Different Positions for α_{i_2}

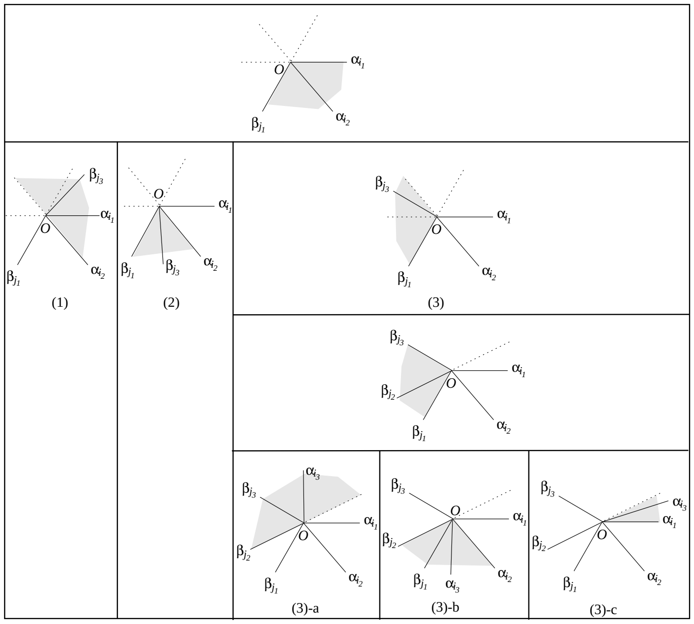
To simplify the proof of the following Theorem 2, we introduce another criteria $d(X)$ applicable to any matching X:

$$d(X) = \sum_{(i,j) \in X} \min((\angle \alpha_i \beta_j)^2, (\angle \beta_j \alpha_i)^2). \tag{7}$$

The following Lemma 2 follows from the proof of Lemma l:

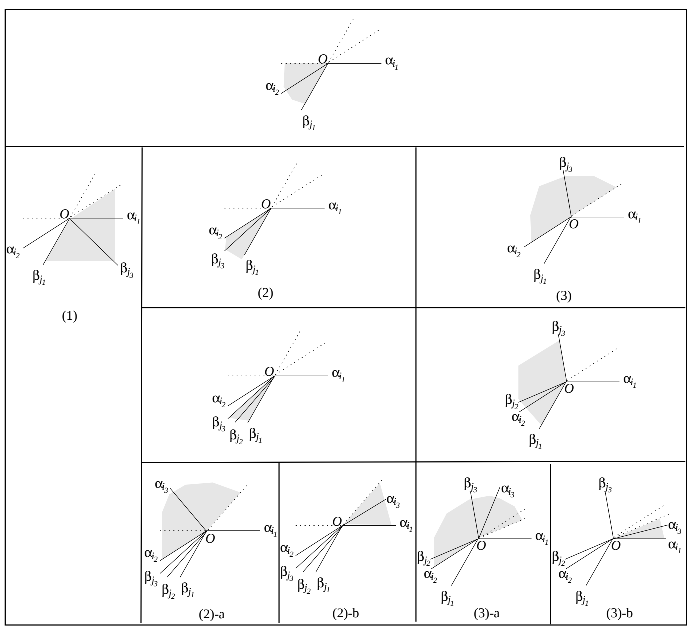
Lemma 2 *$d(X') < d(X)$ is true for each of the X's used in the proof of Theorem 1.*

Theorem 2 *Suppose that α_i, $i = 1, 2, \cdots, n$ and β_j, $j = 1, 2, \cdots, n$ have been rotationally sorted into $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_n}$ and $\beta_{j_1}, \beta_{j_2}, \cdots, \beta_{j_n}$ separately, then the minimum complete matching X could be selected from $\{Y_k | k \in \{0, 1, \cdots, n-1\}\}$, where $Y_k = \{(i_m, j_{(m+k \bmod n)+1}) | m = 1, 2, \cdots, n\}$.*



(1), (2) and (3) are the subcases of $\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1}$. 3-(a), (3)-b and (3)-c are the subcases of (3).

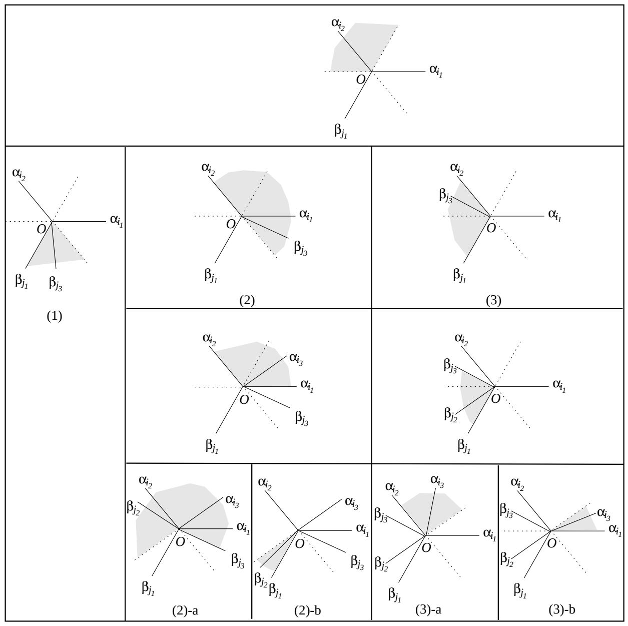
Fig. 4. The Case that $\alpha_{i_1} \prec \alpha_{i_2} \prec \beta_{j_1}$



(1), (2) and (3) are the subcases of $\beta_{j_1} \prec \alpha_{i_2} \prec \bar{\alpha}_{i_1}$. 2-(a) and (2)-b are the subcases of (2), 3-(a) and (3)-b are the subcases of (3).

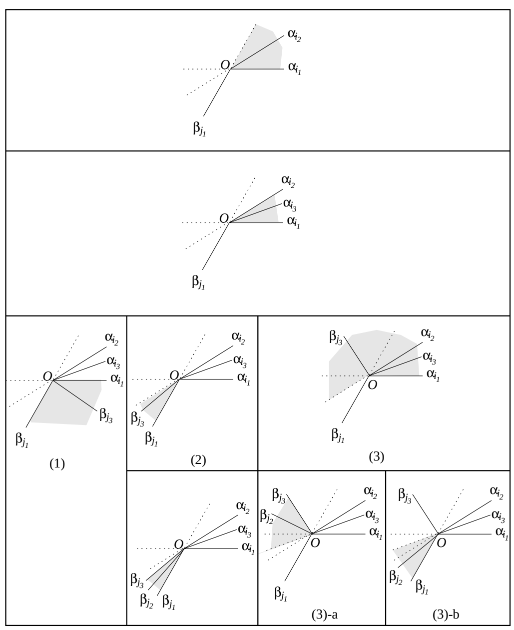
Fig. 5. The Case that $\beta_{j_1} \prec \alpha_{i_2} \prec \bar{\alpha}_{i_1}$

PROOF. Consider any complete matching X_1,which includes a crossover matching. It is easy to show that X_1 could be converted into X_2 such that $w(X_2) \leq w(X_1)$, by eliminating the crossover matching in accordance with Theorem 1. If X_2 has another crossover matching, we could further convert it



(1), (2) and (3) are the subcases of $\bar{\alpha}_{i_1} \prec \alpha_{i_2} \prec \bar{\beta}_{j_1}$. 2-(a) and (2)-b are the subcases of (2), 3-(a) and (3)-b are the subcases of (3).

Fig. 6. The Case that $\bar{\alpha}_{i_1} \prec \alpha_{i_2} \prec \bar{\beta}_{j_1}$



(1), (2) and (3) are the subcases of $\bar{\beta}_{j_1} \prec \alpha_{i_2} \prec \alpha_{i_1}$. 3-(a) and (3)-b are the subcases of (3).

Fig. 7. The Case that $\bar{\beta}_{j_1} \prec \alpha_{i_2} \prec \alpha_{i_1}$

to X_3 such that $w(X_3) \leq w(X_2)$. Thus we could have a sequence

$$X_1, X_2, X_3, \cdots\cdots \tag{8}$$

Noting that Lemma 2 insures that $d(X_1) > d(X_2) > \cdots$, we have $X_t \neq X_s$ for any $t \neq s$. Since the total number of distinct matching is finite, we could finally reach a k at the end of the sequence (8), such that X_k does not include any crossover matching. The conclusion of the theorem is now immediate.

Corollary : Theorem 2 still holds if we replace the definition of $w_{i,j}$ of equation (1) by

$$w_{i,j} = \min((\angle \alpha_i \beta_j)^{1+\delta}, (\angle \beta_j \alpha_i)^{1+\delta}). \tag{9}$$

where $\delta \geq 0$.

This corollary can easily be proved by noting that Lemma 1 still remains valid if we replace the definition of $w_{i,j}$ by equation (9). All the existing works including Aggarwal et al's algorithm are applicable to the very special case of $\delta = 0$ in the corollary.

Theorem 2 shows that an optimal matching X could be obtained by firstly sorting α_i and β_j by any of a sorting algorithm with the complexity of $O(n \log n)$, and then choosing the one satisfying Theorem 2 from among n potential candidates. This can be obviously done with the complexity of $O(n^2)$.

3 Conclusion

This paper generalizes the Gilmore-Gomory matching problem to a more general case where restrictions on angular representations are imposed. Examining the specific travelling salesman problem (TSP) to which the Gilmore-Gomory matching problem ([3]) is applied, it would not be difficult to show that the present the n-Tokyoites' Loop-Line Commuter Problem can also be converted to a generalized version of the specific Gilmore-Gomory's TSP, with a minor modification of Gilmore-Gomory techniques ([3]).

A further study of the specified bipartite matching problem for $|S| \neq |T|$ would be interesting as it finds applications in multitarget tracking problems ([4]) where missing or cluttering sensor data are frequently observed.

References

[1] H. W. Kuhn, "The Hungarian method for the assignment problem," *Naval Res. Logist. Quart.*, vol. 2, pp. 83–97, 1955.

[2] M. L. Balinski and R. E. Gomory, "A primal method for the assignment and transportation problem," *Mgt. Sci.*, vol. 12, pp. 578–593, 1964.

[3] P. C. Gilmore and R. E. Gomory, "Sequencing a one state-variable machine: A solvable case of the traveling salesman problem," *Operations Research*, vol. 12, pp. 655–679, September-October 1964.

[4] S. Blackman and R. Popoli, *Design and Analysis of Modern Tracking Systems*, Artech House, 1999.

[5] E. L. Lawler, *Combinatorial Optimization: Networds and Matroids*, Holt, Rinehart and Winston, New York, 1976.

[6] D. Gale, *The Theory of Linear Economic Models*, McGraw-Hill, New York, 1958.

[7] Alok Aggarwal, Amotz Bar-Noy, Samir Khuller, Dina Kravets, and Baruch Schieber, "Efficient minimum cost matching and transportation using the quadrangle inequality," *Journal of Algorithms*, vol. 19, no. 1, pp. 116–143, 1995.

[8] H. W. Kuhn, "Variants of the Hungarian method for the assignment problem," *Naval Res. Logist. Quart.*, vol. 3, pp. 253–258, 1956.

[9] G. Carpaneto and P. Toch, "Algorithm 548: Solution of the assignment problem[H]," *ACM Trans. Math. Software*, vol. 6, pp. 104–111, March 1980.

[10] D. P. Bertsekas, "A unified framework for primal-dual methos in minimum cost network flow problems," *Math. Programming*, vol. 32, pp. 125–145, 1985.

[11] D. P. Bertsekas, "The auction algorithm: A distributed relaxation method for the assignment problem," *Annals of Operat. Res.*, vol. 14, pp. 105–123, 1988.

[12] M. L. Balinski, "Signature methods for the assignment problem," *Operat. Res.*, vol. 33, no. 3, pp. 527–536, 1985.

[13] D. Goldfarb, "Efficient dual simplex algorithm for the assignment problem," *Math. Programming*, vol. 33, pp. 187–203, 1985.

[14] P. M. Vaidya, "Geometry helps in matching," *SIAM J. Comput.*, vol. 18, pp. 1201–1225, 1989.

[15] R. M. Karp and S. Y. R. Li, "Two special cases of the assignment problem," *Discrete Mathematics*, vol. 13, pp. 129–142, 1975.

[16] Samuel R. Buss and Peter N. Yianilos, "Linear and $O(nlogn)$ time minimum-cost matching algorithms for quasi-convex tours," *SIAM J. Comput.*, vol. 27, no. 1, pp. 170–201, 1998.

[17] R. E. Burkard, B. Klinz, and R. Rudolf, "Perspective of Monge properties in optimization," *Discrete Applied Mathematics*, vol. 70, no. 2, pp. 95–161, 1996.