Stability Analysis of Regional and National Voting Schemes by a Continuous Model

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Abstract

The previous discrete-model-based stability analysis of regional and national voting [5] has been extended to a continuous-model-based analysis in the simultaneous presence of white and concentrated components of noise, reconfirming the previous conclusion that the regional voting with smaller sized regions demonstrates always an improved stability than those with larger sized regions, including the national voting in its limiting case in particular. The conclusion remains valid as long as the weak distribution assumption is valid.

Keywords

decision making, computer vision, mathematical modeling, fault tolerance, national voting, regional voting, noise, stability analysis, continuous model

I. Introduction

VOTING is one of the most popular and important decision making processes not only in our daily social and political activities but also in many scientific studies. The so-called national and the regional voting are the most popular voting schemes widely adopted. In the national voting we select a candidate(s) directly by a simple majority of the entire voting population of the nation while in the regional voting, the winner of the voting is determined by a majority of the winning regions based on the winner-take-all principle where the entire national voting district is divided into smaller regions as into prefectures or states. The US presidential election is a most well-known example of the regional voting and it is worthwhile to know why such a time-consuming election system has a merit over the more simpler national voting system. We want to investigate this from the point of view of stability of decision process against noise which always persists within the voting process in any realistic voting system affecting the final result of decision making. The noise here refers to a factor(s) or their combinations which are capable of enforcing voters to change their mind from one to the other candidates.

In the scientific application area, we have a gap between the theoretical studies of the voting including the game theory, and the application. Most of the theoretical studies emphasize how to choose a

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winning strategy in special voting schemes. Only some experimental works are available showing that some special voting scheme is the more reliable or more stable than the others depending on a specific application [1], [2], [3]. In fact, we must test different voting schemes separately (see [4], for example). To shed more light on the decision making process in scientific applications, we have initiated a general stability analysis of voting against the concentrated noise in an image recognition application by a discrete model [5]. The concentrated noise treated in the image processing can be thought of as representing damaged or polluted images due to ink blotting, leaking sunlight through sunshade, carriers [6], or due to the electric noise at a certain period of times.

The purpose of the paper is to extend the previous discrete-model-based stability analysis [5] to a wider and more realistic situation by a continuous model allowing the presence of both white and concentrated noise components. The present continuous model-based analysis reinforces the previous conclusion of the discrete model-based analysis of [5] confirming that the regional voting with smaller sized regions is always more stable than that with larger sized regions as long as a so-called weak distribution assumption remains valid; accordingly the regional voting with small regions must be more stable than the national voting which constitutes a limit of a largest possible size of such subdivided regions.

We emphasize that the present analysis sheds light on many important applications of scientific studies requiring decision making processes involving image and pattern recognition.

II. The Main Results

A. Simplified Voting Model

1. We consider a simplest possible two candidates voting system comprising candidates A and B.
2. Without losing generality, we assume that A wins the voting over B in the absence of noise.
3. The nation comprises a rectangle area of size $X \times Y$, with X and Y being given by positive real numbers. In a 2-dimensional Cartesian co-ordinate system, the nation is represented in terms of four vertices of $(0,0)$, $(0,X)$, (X,Y) and $(Y,0)$.
4. We partition the nation into a grid of rectangles of size $r_x \times r_y$, such that X and Y are assumed divisible by r_x and r_y independently. We consider the regional voting on the partitioned nation. A dimensionless parameter $r_{\max} = \max\{r_x/X, r_y/Y\}$ is used to denote the relative size of the partitioned voting regions, hereinafter.
5. The voting for candidates A and B over an integrable area $\mathcal{D}$ is defined in terms of the following two integrable density functions $\mathcal{F}_A$ and $\mathcal{F}_B$, where $0 \le \mathcal{F}_A(x,y), \mathcal{F}_B(x,y) \le 1$, and $\mathcal{F}_A(x,y)+\mathcal{F}_B(x,y) = 1$ for $\forall(x,y) \in \mathcal{D}$. Here $\mathcal{F}_A$, $\mathcal{F}_B$ are the density functions of the voting for candidate A and B, representing the ratios of the voters "voting" for candidate A and B respectively both of which are valid within a sufficiently small neighborhood of a point (x,y). Item 2 of the model implies that in the absence of

noise we have $\iint_{[0,X]\times[0,Y]}\mathcal{F}_A dxdy > \iint_{[0,X]\times[0,Y]}\mathcal{F}_B dxdy$. $\iint_{\mathcal{D}}\mathcal{F}_A dxdy$, and $\iint_{\mathcal{D}}\mathcal{F}_B dxdy$ are called the measurements of the votes which candidate A and B get in area $\mathcal{D}$ respectively. The winner of the voting in area $\mathcal{D}$ is decided by a simple majority of the measurements of votes throughout the area.

6. A winner in the national voting is decided by a simple majority of the measurements of votes as computed and measured over the entire nation.

7. A winner in the regional voting is decided by the "winner-take-all" principle namely by a majority of the number of the winning regions, where the winner of a region is decided by the majority voting as computed over the region.

8. An element of noise is called anti-A(or anti-B) noise if the noise enforces an original pro-A voter (or pro-B voter) to change his vote from A(or B) to B(or A). Depending on the types of noise, a probability may be assigned to each element of noise representing the strength of the noise in vote turnover, say, as in [5].

9. Because we are interested in evaluating the lower bound of the voting stability, we consider throughout this paper only the anti-A noise in the analysis which influences the votes cast for candidate A to change to B, implying that the voting density function for B increases by the amount reduced for A in voting density.

It is worth mentioning that the two candidate election system we analyze in this paper is not as restrictive as it looks, because as we show in section III-C.3 the basic principle of the conclusion is still valid for a multi-candidate model involving three or more candidates with the noise included.

B. Definitions

Definition 1: In terms of an integrable density function $\mathcal{N}$ for a noise set mapping from $[0,X]\times[0,Y]$ to $[0,1]$, the effect of an anti-A-noise set in the nation is equivalent effectively to changing the voting density functions $\mathcal{F}_A$ for A and $\mathcal{F}_B$ for B to $\mathcal{F}_A - \mathcal{N}\times\mathcal{F}_A$ and $\mathcal{F}_B + \mathcal{N}\times\mathcal{F}_A$, where $\mathcal{F}_A$ and $\mathcal{F}_B$ are the original density function.

In accordance with item 9 of Section II-A we will be only concerned with the anti-A-noise throughout our analysis. Hence the anti-A-noise will be referred to as the noise hereinafter.

When polluted by a noise set having the density function $\mathcal{N}(x,y)$ in a sufficiently small neighborhood Λ of (x,y), all the votes change to B whether or not they are originally for A or B. We easily find that the effect of the noise set is such that the measurement of votes for A in Λ changes from $\mathcal{F}_A\cdot\Lambda$ to $(\mathcal{F}_A - \mathcal{F}_A\mathcal{N})\cdot\Lambda$, while the measurement of votes for B changes from $\mathcal{F}_B\cdot\Lambda$ to $(\mathcal{F}_B + \mathcal{F}_A\mathcal{N})\cdot\Lambda$.

Definition 2: The white noise is distributed homogeneously within the nation except for a measure 0 set and is characterized by a noise density function $\mathcal{N}$ such that: $\mathcal{N}(x,y)\mathcal{F}_A \propto \mathcal{F}_A - \mathcal{F}_B$ *a.e.w.* [1].

[1]The notation, $P(x,y)$ *a.e.w.* in an integrable area $\mathcal{D}$ implies that the property $P(x,y)$ holds *almost everywhere* such that there exists a set $\mathcal{S}\in\mathcal{D}$ where its Lebesgue measure is 0 (see [7] section 10.3.2.3,p.522 for details on Lebesgue measure) and $P(x,y)$ holds for $\forall\{x,y\}\in\mathcal{D}-\mathcal{S}$.

Definition 3: The set of concentrated noise is defined as the union of a finite number of blocks $\mathcal{S}$ polluted by noise sets each of which is given by

$$\mathcal{N}(x,y) = \begin{cases} 1, & \textit{if } (x,y) \in \mathcal{S} \\ 0, & \textit{otherwise} \end{cases} \quad a.e.w,$$

where $\mathcal{S}$ is a connected open or closed domain. This noise-polluted domain is referred to as a noise block in this paper, with a finite number of blocks having finite length boundaries. The union of all such $\mathcal{S}$ is called the noise concentrated area.

Note that, within the noise block, the effect of the noise is equivalent to changing the voting density for A to 0 $a.e.w.$, with the voting density for B changing at the same time to 1 $a.e.w.$.

Definition 4: In the regional voting, we define the region as a concentrated noise polluted region if a Lebesgue measure of the intersection of the region and the noise concentrated area is not zero. All of the concentrated noise polluted regions are counted as a pro-B region even though the region still remain pro-A.

Definition 5: The measurement of a noise set in an integrable area $\mathcal{D}$ with noise density function $\mathcal{N}$ is defined as $\iint_{\mathcal{D}} \mathcal{N} dxdy$.

The measurement of noise represents the "number" of votes polluted by the noise in the area.

Definition 6: When polluted by a noise set in an integrable area $\mathcal{D}$ having noise density function $\mathcal{N}$, $\iint_{\mathcal{D}} \mathcal{N}\mathcal{F}_A dxdy$ is called the measurement of the noise-polluted (to be strict, anti-A-noise-polluted) votes in $\mathcal{D}$. We always use $\aleph$, $\aleph_c$ and $\aleph_w$ to denote the measurements of noise polluted votes, concentrated noise polluted votes and white noise polluted votes over the entire nation respectively .

Note that all of the noise polluted votes change to votes for B whether the original votes are for A or for B with a given probability (see Item 8 of ll-A for details on assigning probabilities).

C. Theorems

Assumption 1 (Weak Average Distribution Assumption) The Weak Average Distribution Condition prevails in each of the regions in the regional voting, where we call an area $\mathcal{D}$ following Weak Average Distribution Condition Conditions provided that $\iint_{\mathcal{D}}(\mathcal{F}_A - \mathcal{F}_B)dxdy \cdot \iint_{[0,X]\times[0,Y]}(\mathcal{F}_A - \mathcal{F}_B)dxdy > 0$.

The assumption of the Weak Average Distribution Condition in an area implies that the area retains the same dominating relationship of the measurements of the votes among the two candidates of the nation.

This assumption implies that in the absence of noise, if candidate A (or B) wins in the national voting, so does candidate A (or B) in each of the regions.

Theorem 1: In the presence of both components of white and concentrated noise, the national voting is capable of preserving the original voting result *if and only if* the measurement of noise-polluted votes is less than $\frac{\iint_{[0,X]\times[0,Y]}(\mathcal{F}_A - \mathcal{F}_B)dxdy}{2}$.

Proof: The conclusion comes easily by computing how much we can accommodate noise before the reversal of decision takes place for the national voting where a simple majority principle applies. ■

We note that the assumption of the Weak Average Distribution Condition above plays no role in the theorem on national voting.

Lemma 1: Let $\mathcal{L}$ be a closed curve within the nation, $|L|$ the length of the curve, and I the total number of the rectangular regions which intersect with the curve. We then have $I \leq 4 \times \left\lceil \frac{|L|}{\min\{r_x, r_y\}} \right\rceil$.

Proof: The above conclusion is immediate when $I \leq 4$. Let us now consider the case of $I > 4$. We regard $\mathcal{L}$ as a directed curve by arbitary provide a direction alone the curve, and arbitary fix one point at the curve as the starting point. For any 4 rectangles which insersect the curve $\mathcal{L}$, we can topolically logically sort them into R_1, R_2, R_3, R_4, such that the directed curve first goes into R_1 at a point a, then passes through R_2 whether by way of some rectangles or directly, and similarly passes through R_3 whether by way of some rectangles or directly and goes into R_4 leaving from $R_1 \cup R_2 \cup R_3 \cup R_4$ to other rectangles at a point b. By examining figure 1, we see that the length of the curve $\widehat{ab}$ is not smaller than the $\min\{r_x, r_y\}$.

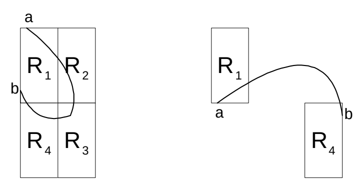


Fig. 1. The Segment $\widehat{ab}$ of the Curve between R_1 and R_4

Let us topologically list the rectangles that are intersected by the directed curve by sequence, we observe from above that that each 4 sequentially neighboring rectangles in the list will occupy a segment of the curve no shorter than $\min\{r_x, r_y\}$. Thus the conclusion is immediate now. ■

The following Lemma 2 follows immediately from Lemma 1.

Lemma 2: Let S_p be the union of all the concentrated noise polluted regions. If the noise concentrated area has n boundary curves with total length L, we have $|\mu S_p - \mu \mathcal{S}| \leq 4 \left(\left\lfloor \frac{L}{\min\{r_x, r_y\}} \right\rfloor + n \right) \cdot r_x r_y$, where $\mathcal{S}$ is the noise concentrated area, and μ is the 2-dimensional Lebesgue measure.

Given the density function of white noise as $\mathcal{N}$, $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2}$ immediately implies that $\mathcal{N}(x,y)\mathcal{F}_A < (\mathcal{F}_A - \mathcal{F}_B)/2$ *a.e.w.*, ensuring the dominating position of candidate A in each region. This is so because $\mathcal{N}(x,y)\mathcal{F}_A \propto \mathcal{F}_A - \mathcal{F}_B$ *a.e.w.* from Definition 2. Together with the above Lemma, we have

Theorem 2: A sufficient condition for the regional voting to preserve the original candidate selection prevailing in a noise free condition is $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2}$ and $\mu \mathcal{S} < \frac{XY}{2} - 4 \left(\left\lfloor \frac{L}{\min\{r_x, r_y\}} \right\rfloor + n \right) \cdot r_x r_y$. The sufficient condition simplifies to $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2}$ and $\mu \mathcal{S} < \frac{XY}{2}$F for a sufficiently small region size so that $\max\{r_x, r_y\}$ is sufficiently small.

D. Corollarys

In addition to the Weak Average Distribution Assumption of the preceding subsection II-C, the following Strong Average Distribution Assumption within *the noise concentrated area* is useful whereby we assume that the same average ratio in the measurements of votes for candidate A and B as observed over the nation prevails within the area in the absence of noise. Mathematically we write:

Assumption 2 (Strong Average Distribution Assumption) The Strong Average Distribution Condition prevails in the noise concentrated area of the nation, where we call an an area $\mathcal{D}$ following Strong Average Distribution Condition provide that in the absence of noise $\frac{\iint_{\mathcal{D}} \mathcal{F}_A dxdy}{\mu \mathcal{D}} = \frac{\iint_{[0,X]\times[0,Y]} \mathcal{F}_A dxdy}{XY}$, where μ denotes a 2-dimensional Lebesgue measure.

The following Corollary 1 follows from the assumption:

Corollary 1: The national voting will preserve the original candidate selection if and only if

$$\aleph_c < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2} \text{ and } \aleph_w < \frac{1}{1 - \aleph_c / \iint_{[0,X]\times[0,Y]} \mathcal{F}_A dxdy} \cdot \left(\frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2} - \aleph_c \right).$$

Proof: According to assumption 1,

$$\aleph_w = \iint_{[0,X]\times[0,Y]-\mathcal{S}} \mathcal{N}_w dxdy \cdot \frac{XY}{XY - \mu \mathcal{S}},$$

where $\aleph$ is the union of all the noise polluted votes, $\mathcal{S}$ the noise concentrated area and μ the 2-dimensional Lebesgue measure. We observe that:

$$\aleph = \iint_{[0,X]\times[0,Y]-\mathcal{S}} \mathcal{N}_w \mathcal{F}_A dxdy + \iint_{\mathcal{S}} \mathcal{N}_c \mathcal{F}_A dxdy$$

The proof follows immediately from theorem 1. ■

Corollary 2: The sufficient condition for the regional voting to preserve the original candidate selection prevailing in a noise free condition is $\aleph_c < \left(\frac{1}{2} - \frac{4\left(\left\lfloor \frac{L}{\min\{r_x, r_y\}} \right\rfloor + n\right) \cdot r_x r_y}{XY} \right) \iint_{[0,X]\times[0,Y]} \mathcal{F}_A dxdy$ and $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2} \mathcal{F}_A$. Here the noise concentrated area $\mathcal{S}$ has n boundary curves with the total length of L. It is easy to see that when the size of regions $\max\{r_x, r_y\}$ is sufficiently small with respect to X and Y such that the parameter $r_{\max} \to 0$, the sufficient condition simplifies to $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]} (\mathcal{F}_A - \mathcal{F}_B) dxdy}{2}$ and $\aleph_c < \frac{1}{2} \iint_{[0,X]\times[0,Y]} \mathcal{F}_A dxdy$.

E. Example

The stability diagram of figure 2 is prepared from these theorems and it is most instructive to demonstrate how the stability of the regional voting is improved over the national voting scheme against a combination of two components of noise; both the white noise and the concentrated noise are represented by the y-axis and x-axis respectively. The stability diagram is given for the case of

$\mathcal{F}_A = 0.51$, $\mathcal{F}_B = 0.49$ and $X = Y = 100$. To facilitate the comparison all quantities in the diagram are normalized by the entire population ($XY = 10000$). Stability boundary A-B of figure 2 comes directly from the two equations of Corollary 1 of the national voting and the stability boundary A-C-D from the last two of Corollary 2 of the regional voting.

1. The area O-A-B-O constitutes a stable domain for the national voting;
2. The area O-A-C-D-O is a stable domain for the regional voting when the size of region is sufficiently small as $r_{\max} \to 0$;
3. The area A-B-G-F-E-A is an unstable domain for the national voting;
4. The area A-C-D-G-F-E-A is an unstable domain for the regional voting when the size of region is sufficiently small.

Here, an area D is called *a stable domain* (*an unstable domain*) for the voting scheme, if the voting scheme is stable (unstable) against the concentrated noise with measurement $\aleph_c$ as well as the white noise with measurement $\aleph_w$ such that $(\aleph_c, \aleph_w) \in D$.

The diagram clearly demonstrates the stability of the regional voting over the national voting because a much wider stable domain exists for the regional voting than the 'single-partitioned' national voting if partitioned regions are sufficiently small.

In figure 2, the boundary C-D gives a limiting case of vanishingly small regions namely as $r_{\max} \to 0$. If $r_{\max}$ is not too small, the boundary C-D now moves to the left side such as C'-D' which corresponds to the case of $L = 75$, $n = 2$ and $r_l = r_w = 10$ ($r_{\max} \approx 0.1$).

We will come back to this topics in item 3 of subsection III-A.

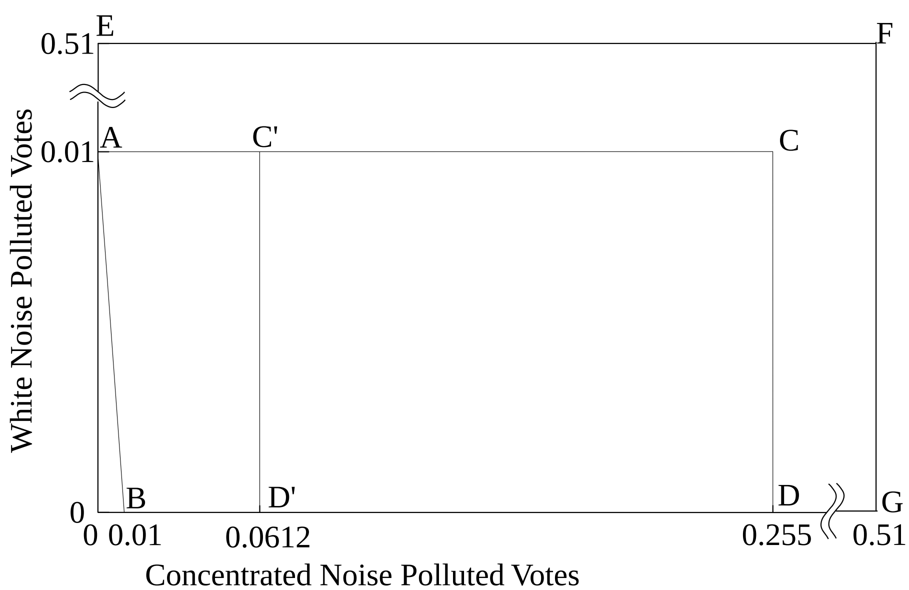


Fig. 2. Stability Diagram for $\mathcal{F}_A = 0.6$, $\mathcal{F}_B = 0.4$

III. Conclusion, Experiment and Discussion

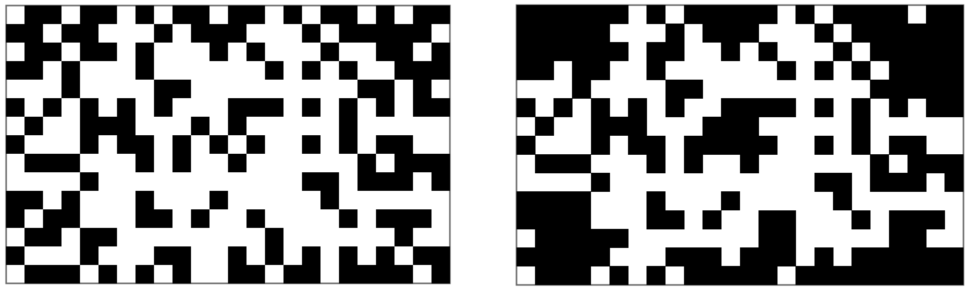
A. Conclusions

1. The theorems clearly show that the regional voting is the more stable and hence robust of the two voting schemes against two components of white and concentrated noise.
2. It is easy to see that in the regional voting, the larger the number of the regions of partitions, the more noise we are able to accommodate without changing the original voting results provided that the Weak Average Distribution Assumption remains valid in the partitioned regions.
3. In proving Theorem 2 and Corollary 2, only those regions entirely *free from noise* are included as a pro-A region. Undoubtedly many of those regions counted as pro-B still remain pro-A in practice with an adequate margin to a critical point of decision reversal. Thus, we have every reason to believe that the regional voting can accommodate even more noise as implied by the present analysis (Theorem 2 and Corollary 2).

B. Experiment

We present a simple experiment here elucidating how we recognize a white-black mixed flag either as a white dominated or a black dominated flag in the presence of noise(see figure 3 for illustration where the cells in the figure denote a smallest unit of a "pixel"). The size of the "nation" is 24×15 cells and we partition the flag into 40 and 18 regions comprising 3×3 or 4×5 partition cells which we believe are not either too large nor too small relative to the size of the nation. Suppose that originally a white-dominated flag is given as in figure 3(1). This is confirmed easily by both the national and regional voting because in the global voting, "White" gets 207 votes while "Black" 153 votes; on the other hand, based on a 4×5 regional partitioning vote counting ($r_{\max} = 0.17$), "White" wins in 12 regions while "Black" does in 4 regions, with 2 regions tied up by equal "White" and "Black" votes. For a 3×3 partition of 40 regions ($r_{\max} = 0.125$), "White" wins in 28 regions while "Black" does in 12 regions in regional voting. To simulate introducing the combined elements of white and concentrated noise, we randomly choose 7 blocks of size 5×5 and 7 cells in figure 3(1) (we allow overlaps of blocks and cells), where each "White" cell within the union of the blocks and cells has a chance to be transformed to "Black" with a probability of 0.7. As a result, when anti-white-noise polluted, 36 "White" pixels are changed to "Black" transforming figure 3(1) to figure 3(2). By counting, we see that after the noise addition, the national voting will reverse the results of the candidate selection from "White" to "Black" dominated because this time "Black" gets 189 votes while "White" only 171 votes in national voting. But, by the regional voting having the size of 4×5 cells, the original selection of a "White" dominated flag still remains valid, because this time "White" wins in 9 regions while "Black" does so in 6, with another 3 regions tied up. This stability margin increases for a smaller partitioning of 3×3 sized regions, where "White" wins in 25 regions while "Black" does in 15 in the regional voting increasing the stability margin thus confirming our theory. Noting that $9/6 < 25/15$, we confirm that the smaller regions give a better stability margin, agreeing with the result of Theorem 2.

We emphasize that the same argument follows even if the original and the polluted flags interchange between figure 3(2) and figure 3(1), implying that the regional voting is the more stable in the sense that the regional voting schemes for the two flag select the same candidate "White".



(1) (2)

Fig. 3. "White" or "Black" Dominated?

To further demonstrate the validity of the experiments so that our example is not artificially constructed to "fit" the theorems, we have shifted the partitions vertically as well as horizontally by one cell up to the total of $3 \times 3 = 9$ different partitions for the regional voting with the regions of size 3×3. To do this it is helpful to regard the pair of the opposing edges along the outer boundary of the flag as glued together so that the other boundaries join onto the other boundaries. A careful examination of the voting results of 9 different regional voting figure 3(2) shows that 7 out of 9 partitioning retain the "White" dominating flag, with only 2 exceptions in each of which "White" only wins in 18 regions out of 40 regions.

C. Discussion

C.1 Weak Average Distribution Assumption

The effectiveness of the assumption can be discussed in the following two different cases seperately.

- In the image processing application, the assumption may not be valid in certain region, say, $\mathcal{R}$, where the votes for candidate A exceed distinctly those in all the other regions where the weak assumption holds. This is certainly true if the property of the region $\mathcal{R}$ comes from an entirely different object other than the main object we try to recognize. This often happens if the region $\mathcal{R}$ constitutes some background objects entirely different from the main object of our recognition. Evidently the region $\mathcal{R}$ should not be included in "voting" for recognition in the first place.
- In the political voting application, this assumption is called "consistency" criterion which requires that the separate elections in arbitrarily divided consistent regions should result in selecting the same candidate which should also agree with the candidate to be elected by the national voting. Otherwise,

the voting system may be manipulated through the establishment of strategically configured election districts [8].

We believe that this assumption should hold, whether in political or scientific applications.

C.2 Strong Average Distribution Assumption

The stability affordance of the regional voting over the national voting does not depend on the strong average distribution assumption. To see this, it is sufficient to note that the national voting can accommodate a measurement $\frac{\iint_{[0,X]\times[0,Y]}(\mathcal{F}_A-\mathcal{F}_B)dxdy}{2}$ of the *union* of white noise polluted votes and concentrated noise polluted votes (Theorem 1), while the regional voting can accommodate that of white noise polluted votes as well as an additional concentrated noise simultaneously (Theorem 2).

The strong average distribution assumption is introduced to help provide a quantitative evaluation of the stability diagram of the regional voting and the national voting. We believe that this assumption remains valid because the noise is distributed random in principle.

C.3 Multiple-candidate voting

We show below that the major conclusion of the stability analysis always holds even if there is more than two candidates, although the special coefficients of equations in the theorems may change. We show this by the following theorem by considering only 3 candidates and the noise is *anti-A* noise only.

Theorem 3: Suppose the density functions of the voting for candidates A, B and P are $\mathcal{F}_A$, $\mathcal{F}_B$ and $\mathcal{F}_P$ respectively, and $\iint_{[0,X]\times[0,Y]}\mathcal{F}_A dxdy > \iint_{[0,X]\times[0,Y]}\mathcal{F}_B dxdy > \iint_{[0,X]\times[0,Y]}\mathcal{F}_P dxdy$, $\iint_{[0,X]\times[0,Y]}\mathcal{F}_A dxdy - \iint_{[0,X]\times[0,Y]}\mathcal{F}_B dxdy < \iint_{[0,X]\times[0,Y]}\mathcal{F}_P dxdy$ [2], $\mathcal{F}_A+\mathcal{F}_B+\mathcal{F}_P=1$ *a.e.w.*. The anti-A-noise with density $\mathcal{N}$ in an area $\mathcal{D}$ can be defined as follows which effectively transforms the voting density function of B and P to $\mathcal{F}_B+\mathcal{N}\cdot\mathcal{F}_P$ and $\mathcal{F}_P-\mathcal{N}\cdot\mathcal{F}_P$ respectively. Then when the size of a region is sufficiently small, $\aleph_w < \iint_{[0,X]\times[0,Y]}(\mathcal{F}_A-\mathcal{F}_B)dxdy$ and $\aleph_c < \frac{1}{2}\iint_{[0,X]\times[0,Y]}\mathcal{F}_P dxdy$ is a sufficient condition for the regional voting to maintain the original candidate seletcion, while the necessary and sufficient condition for the national voting is $\aleph_c < \iint_{[0,X]\times[0,Y]}(\mathcal{F}_A-\mathcal{F}_B)dxdy$ and $\aleph_w < \frac{\iint_{[0,X]\times[0,Y]}(\mathcal{F}_A-\mathcal{F}_B)dxdy-\aleph_c}{1-\aleph_c/\iint_{[0,X]\times[0,Y]}\mathcal{F}_P dxdy}$.

D. Further Works

We believe that the region as used in the context of the present analysis needs not be constrained to the geometrical context as we use in the analysis. It can be considered as a society of people sharing the same race or sharing the same interest in political area, or as a set of information having the same frequency or the same features in speech recognition and image processing[3]. The application in these

[2] This assumption indicates that it is possible to reverse the final candidate selection when certain percentage of the votes cast for P change to B. The situation reminds the case of Mr.Perot in the 1992 US Presidental Election as 19.0%, 37.7% and 43.3% of the popular votes went to Ross Perot, George Bush and Bill Clinton.

[3] Under these situations, we can certainly arrange the society or the information in such a way that those with similar race/interest background, or same frequency/feature are grouped into jointed or almost jointed areas closely following the geometric regions when regional voting takes place

fields is of great interest.

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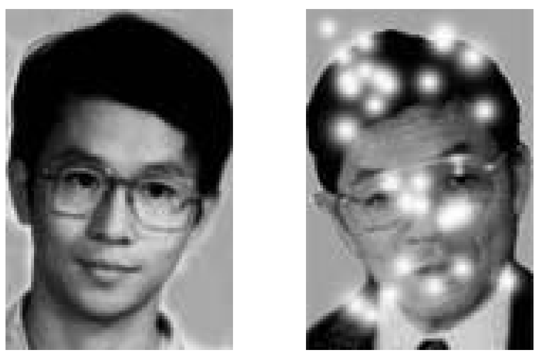
Appendix: Regional Voting on PCA-based Facial Recognition Application

Although the conclusions on national and regional voting schemes are based on a proof of the simple majority voting principle by "pixel-by-pixel"-based counting, we believe that the present theory remains valid on a more general condition. A conjecture will be made here that the validity of the present conclusions extend to a more popular decision making process such as the dimension-reducing schemes of PCA (Principle Component Analysis)[9] or Gabor transformation.

As an example, we select the principal component analysis (PCA) as a fundamental national voting scheme, which is a well known dimension-reducing scheme leading to the famous eigenface method of [9]. The corresponding regional voting method is thus implemented first by dividing the whole two dimensional picture of rectangle frame into smaller regions (of equal size in our analysis). Within each region we make use of the PCA method in decision making and we make the final decision of the whole picture in accordance with the simple majority principle in the number of the winning regions where the winner gets all the votes of the region. We have used the images of 16 people as shown in figure 4(1), and the noise polluted versions as shown in figure 4(2).



(1)Original Image *(2) Noise Poluted Image*

Fig. 4. Typical Noise Contaminated Images

Recognition rates of the experiments are compared in figure 5 illustrating the results of 1-region (corresponds now to the original national voting) up to 16-, 32-, 64-, $\cdots$ regional voting. The figure shows clearly that the regional matching always gives a better recognition rate than the national matching, and that the smaller the regions we choose, the higher recognition rate we will have for the regional matching. As it is mentioned in the paper, the conclusion holds *under the weak average distribution assumption* . When the regions divided are too small, say, each region being of size 3×2 pixels for 1600 regions in this example, the recognition rate will decrease and deteriorate. We believe that, under this situation the Weak Average Distribution Assumption is not valid any more.

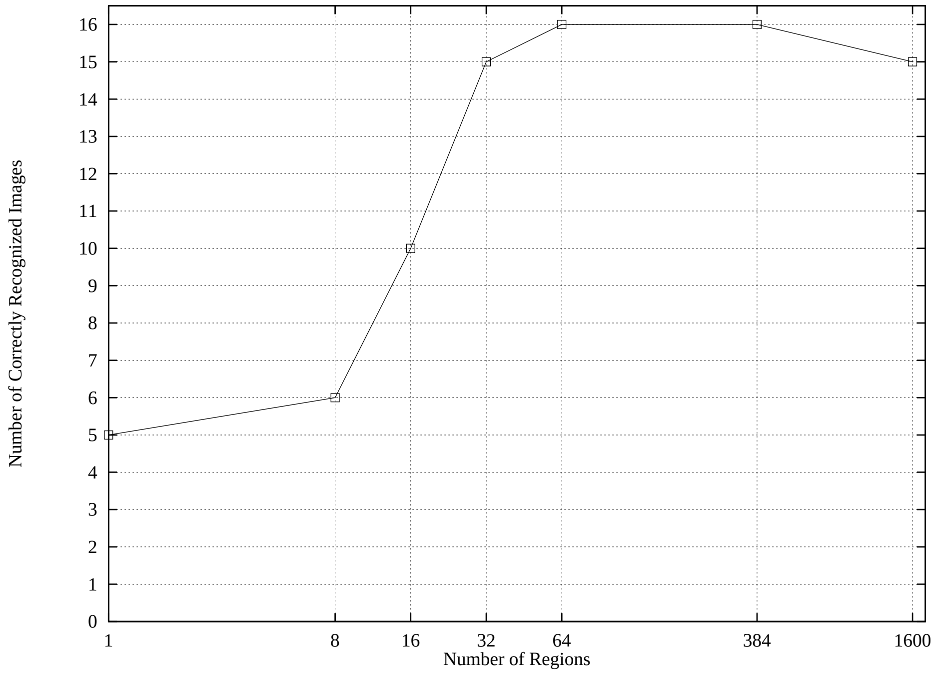


Fig. 5. Recognition Rates of Each Regional Voting Scheme and National Voting

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