Approximately Fast Algorithm for Covering and Packing Problems

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Abstract

Packing and Covering problems are the problems with important applications and can be very simply described. But from the view point of computation, they belong to the class of most difficult problems in computer area. This paper introduces two schemes for solving the problems.

Keywords

Packing Problem, Covering Problem, Fast Algorithm

I. Introduction

The Packing Problem can be described as follows: Given a container with defenite size and shape in the space, and some objects with given sizes and shapes and associated with a weight, it is required to place some of of the objects with a maximal sum of weights into the container without overlap.

The Covering Problem can be described as follows; Given a certain number of objects in the space, and some containers with given sizes and shapes and associated with costs, it is asked to place a number of containers with minumam sum of costs to cover all the objects.

Packing and covering problems has its obviouse applications in the areas of locting emergency facilitis and VLSA chip manufacture and other field.

For example, a silicon wafer is filled with 16M RAM chips, some of them are defective (figure 1(1)). $2 \times s$ non-detective chips can be wired together to produce 64M RAM chips, In order ot maximize the mamnufacture, the problem is just a packing problem of packing 2×2 squares in a grid of unit squares with some of which are defective (figure 1(2)). Another example comes from the agriculture application, each agriculture airplane can be used to distribute the madicines for certain round areas with a reasonable radius, lots of cases ask us to locate a minmum number of planes for cerntain areas.

Both packing and covering problems and covering problems has its determining version, as whether the container can be placed in a certain number of certain sized and shaped objects without overlap, and whethere the the certain a mount of containers can cover all the objects in the space. It is obvious that if determination versions of the two problems

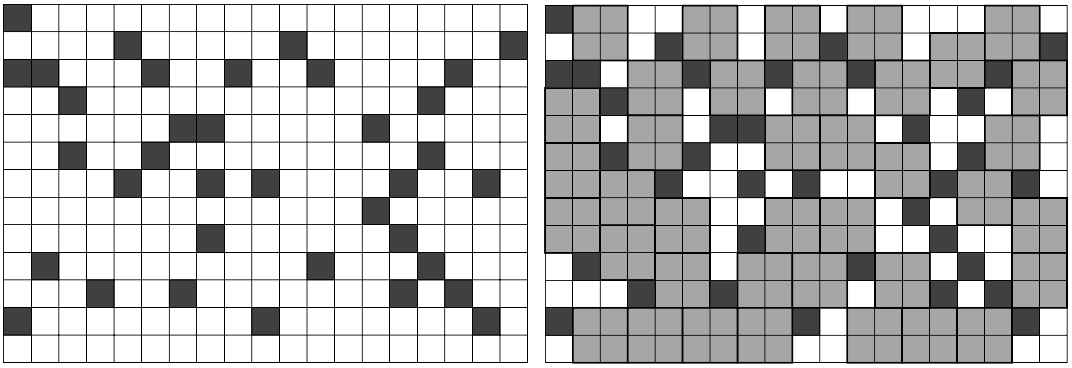


Fig. 1. Chip Cutting

have fast algorithms, the original versions can also be solved quite fast by employing binary searching stratage.

Unfortunately, theoratical computer science had proved that both the packing and covering problems are NP-complete ([3]), even in very simple cases, say the objects in the packing problem and the containers in the covering problem are of same size and shape.

For example, the covering problem of which all the objects are points in the plane, and the containers are disks with same redius, is still a NP-complete problem. And the above mentioned chip cutting problem of 2×2 sized chips is also an NP-complete. When a problem is proved to be the NP-hard (or NP-complet), it is sure that we can not find a fast algorithm to find the optimal solution unless NP=P (The problem whether P=NP is a very difficult problem in computer science field, and it seems that the problem has a negtive answer.

Nowadays, what we could do is trying to find approximately fast algorithms for these problems. What we mean approximately fast is : either the algorithm is fast but always find the "less" optimal solution, or the algorithm can find the optical solution in lots of cases but failed in some rare cases.

In this artical, we introduce two methods for the Packing and Covering problems by simple examples.

II. Shifting Method

Shifting Method for the packing and coveering problem is trying to find the less optimal solutions in short time. The method was independently discovered in [1] and [2].

Given an fixed area $\mathcal{C}$ of in the space of fixed dimensions $d_1, d_2, \cdots, d_n$. Suppose we have strategy λ_i to find an satisfiable solution of $\mathcal{C}$ bounded within any two paralle sup-plane $d_i = x$ and $d_i = y$ such that $|y - x| = lD_i$, where D_i is a fixed real number associated with dimension d_i. Shortly we define $d_i = x$ is on the left of $d_i = y$ if $x < y$.

The shifting strate is described as follows:

1. Fixed a partition of $\mathcal{C}$ by paralle sup-planes with distances lD_i.
2. Perform step 3 - 5 l times
3. Shift all the paralle lines from left to right by a step of D_i
4. Apply strategy λ_i with in each of two neibor supplanes to find the satisiable sulutions there.
5. Sum up the sulutions of each of the ships to be the solution of this patition.
6. Within all these l ways of partitioning find the best solution as the solution for space $\mathcal{C}$.

Recursively apply the shifting stratage, we can costructed a method to find an optimal solution in a space $L_1 \times L_2 \times \cdots \times L_n$as follows: begin an enumeration method λ_0, which can always find the best solution in a super rectangle area $dD_1 \times dD_2 \cdot \cdots \cdot dD_n$, we construct the stratage λ_1 on any rectangle area of size $L_1 \times dD_2 \times dD_3 \times \cdots \times dD_n$; By applying shifting stratage using the stratage λ_1, we construct the stratage λ_2 on any rectangle area of size $L_1 \times L_2 \times dD_3 \times \cdots \times dD_n$; ... .

So, we constructed the stratage of soving the problem in the area $L_1 \times L_2 \times \cdots \times L_n$ by the recursively using shifting stratagy and the begining enumerate method λ_0.

A. Shifting method for covering problem

We discuss the shifting method for finding the a minimum number of diskes with diameter D to cover n given points in the plane.

The method is coming from the following two algorithm:

Algorithm 1:

1. Fix a partition of the involved area by paralle planes with distance lD, repeat the step 2 to 4 l times;

2. Shift the partition from left to right by a distance D;

3. Apply the algorithm 2 to find the best approximate solution with in each pair of the naibor paralle planes;

4. Sum together of the solutions of each "strips", to form the solution for the covering problem in this patition;

5. Compare the results of all these l partitioning, find the most best approximate solution.

Algorithm 2: (Find the best solution in a "strip")

1. Fix a partion of the strip by paralle lines with distance lD, repeat the step 2 to 4 l times;

2. Shift the partition from low to up by a distance of D;

3. By using enumerate mathod, find the best solution with in each squares of size $lD \times lD$;

4. Sum together all the best solutions in each squares to form the solution for the covering problem in this patition;

5. Compare the results of all these l partitioning, find the most best as the solution for this strip.

Notice that for any of a fixed partitioning of the strips in to squares of size $lD \times lD$, the sum of optimal solutions in each of the squares is less then the optimal solution of the whole strip plus the number of the disks in the optimal solution which cover points in two adjancent squares. By noting that if a disk the optimal selution in a strip could cover points in two adjacent squares in one partioning, it could not cover points in two adjancent squares in another partioning (see figure 2). Thus, by applying the pegion hole princeple, we know that the solution of the algorithm 2 for any strip, which is the best of l diffierent partitioning, is less then $1 + \frac{1}{l}\times$ the optimal solution. And it is not difficult to conclude that the the solution the covering problem in the plane is $(1 + \frac{1}{l})^2\times$ the optimal solution.

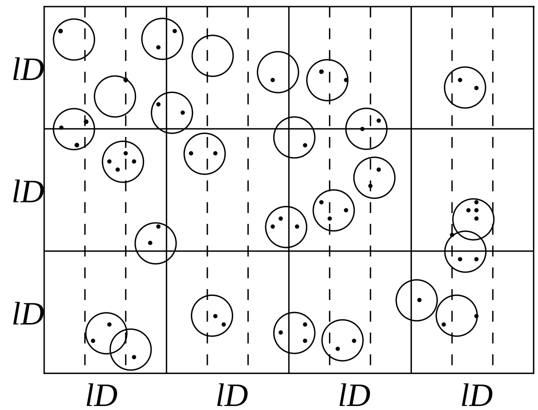
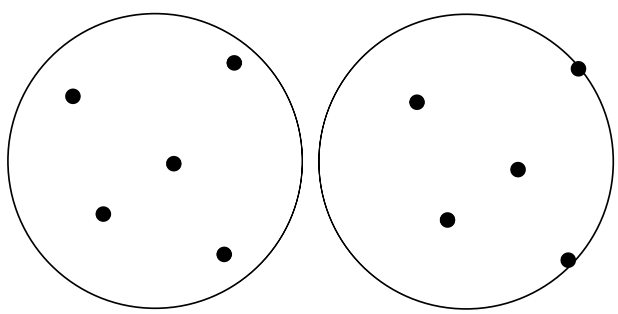


Fig. 2. The Shifting Method for Covering Problem

Notice that, exluded the diskes that cover the isolated points in the space, of which the distance to the nearest points is larger than D (we need a disk to cover each of them), we can suppose that any of the disks in the solution pass through two points in its border. It is because, if a disk in the optimal solution dose not pass through two points in its border can be moved to a position to pass through two points, by keeping the modified solution optimal(Figure 3).



(1) Original Position (2) New Posion

Fig. 3. The Position of A Disk

Also notice that we only need to locate less than $(l \cdot \sqrt{2})^2$ disks of diameter D to cover all the points in the square of size $lD \times lD$.

So, enumerately find the optimal solution in such a square with $\bar{n}$ of points, we only need to enumerate less then and for all the points, supose $\bar{n}$ of points, in the square, there is at most $(l \cdot \sqrt{2})^2$ disks in less than $2\begin{pmatrix} \bar{n} \\ 2 \end{pmatrix}$ posible positions. Thus it can be done within polynomial time. From the point of view of computer science, it can be considered reasonable.

B. Shifting stratage for packing Problem

We give the example of packing a maximal number of $k \times k$ squares into an area that is given by n unit squares on a rectilinear grad.

The algorithm can be exactly described as the algorimths 1 and 2, except that the enumerate method there is now becoming enumeration method to pack a maximal number of $k \times k$ squares into the $lk \times lk$ area.

By the samilar analysis, we can conclude that the algorithm can get the results with more than $(1 - \frac{1}{l})^2$ of optimal number of squares.

To observer that we can at most packing l^2 of $k \times k$ squares into the area given by $\bar{n}$ unit squares on a squared area of size, and there is at most $\bar{n}$ posible position there. So, the local enumerate method can be solved within $O(\bar{n}^{l^2})$ steps, which is also a polynomial complexity. And thus, the whole packing problems can be solved within polynomial time.

III. Quasiphysical method

Quasiphysical stratage comes from the simulating the natural phenomina. The quasiph-physical method is best suitable in the determinination version of packing and covering problem. The quasiphisical method for packing problems is inspired by the picture of movement of smooth elastic objects driven by elastic force, while the method for covering problems inspired by the movement of magnetic objects driven by magnetic forces.

A. Packing Problem

We explain the method for the packing problem as follows: In a given container, it is hoped that N objects of variou know shapes and sizes should be placed. The closed boundary of the bounding container and the objects are all impenetrable rigid solids. If it is impossible to put alll these objects in the container, then it is requared that a judgement to that effect should be made, If it is possible to place the objects in it, it is requared that the position and direction of each object whould be given. A familiar 2-dimensional example is how to put the building blocks made into a mess by children back into the box.

We first suppose the objects and container are elastic, and squeez the objects into the container. Then, simulate the motion resulting from the elastic forces to restore the objects and the container to their original shapes and sizes. Consider the exterior of the container as an elastic solid in which there exists a cavity and regard the solic as the the 0-th object. If the original packing problem has a positive solution, the simulating may find a "state" where the sum of the elastic energy between each pair of objects is zero.

For any two objects O_i and O_j, the elastic energy is defined as

$$U_{ij} = \begin{cases} 0, & \text{if } L_{i,j} \geq 0, \\ L_{i,j}^2, & \text{if } L_{i,j} < 0. \end{cases}$$

where $L_{i,j}$ is the distance between the two objects, which is defind as follows: When the measure of the intersection of object i and j is great than zero, $L_{i,j}$ is a certain negative number whose absolute value is equal to the length traversed by objects i and j that depart from each other in the direction of the line between their centers until the measure of the intersection is zero; when the measure of the intersect of the two objects is zero, $L_{i,j}$ is a certain non-negtive number, whose numerical value is equal to the length traversed by objects i and j that appriach each other in the direction of the line between their centers until the measure of the intersection is 0^+. (Fig.4).

Thus, the total elastic potential energy of a system composed by the N objects and a container is defined as

$$U = \sum_{i=0}^{N-1} \sum_{j=i+1} NU_{ij}$$

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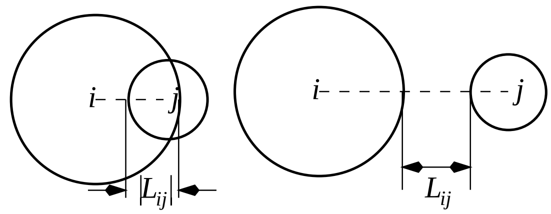
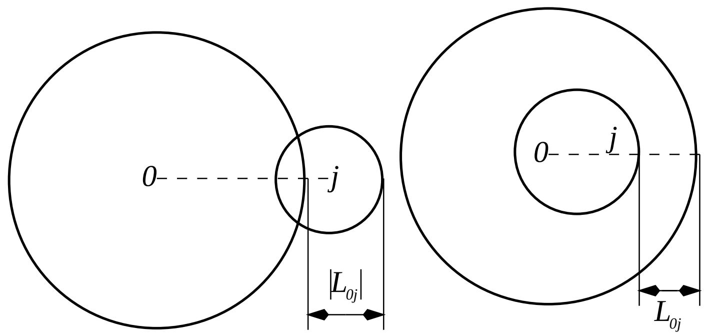


(1) The distance between two objects



(2) The distance between the container and an object

Fig. 4. Difinition of Distance

Form that on, what we need to do is try to find the minmum of the above energy function. This can be down by traditional method of minimizating multidimentional function, or even by the help of simulated annealing stratage.

With this way of simulating by methmatical method, we can solve the packing problem quite fast in most of the cases when the space is not too intension. A simulating picture of the mothod can be see from figure 5. Details of the quasiphycal method for the packing problems can be found in papers [4]

B. Covering problem

We explain the method for the covering problem as follows: given a finite number of points (stationary) on a plane are known and, on the other hand, we have at hand a finite number of disks with known radius, it is hoped to arrange the positions for the disks so that all these points will be covered up.

We first suppose that the plane is in a horizental positon and absolutely smooth and the finite number of points are particles with unit positive electromagnic and fixedly embedded

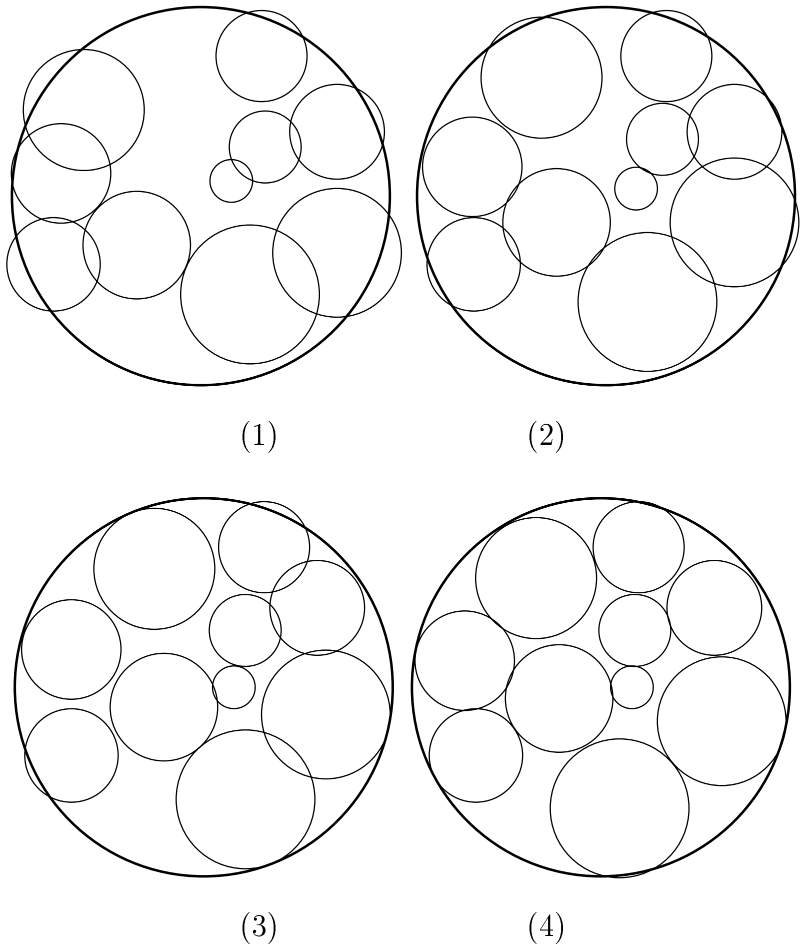


Fig. 5. The Process of Packing by the force of Elastic Forces

in this plane, and the disks are also absolutely smooth, with unit negtive electomagnic that is uniformly distributed on the circumference. Then souppose the the particals will attract the disks to cover themselves owing to the magnic forces between particals and disks. The solution is found if each of the particals is covered by at least one disks. For this purpose we also suppose that when a partical is covered by one disk, the magnic force between this particals and other disks is screened out.

For a pair of partical and disk, the potential energy between them can be defined similar as it was in Packing problem. And thus we computer the energy and minimize it by traditional minimization methods (e.g., [7]).

With this simulting by methematical method, we can solve the covering problems quite fast in most of cases, when the number if the disks is not so limited.

Details of the qusiphysical method for covering problems can be found in paper [5], [6]. A process of for the solving the covering by quasiphysical method is shown in figure 6.

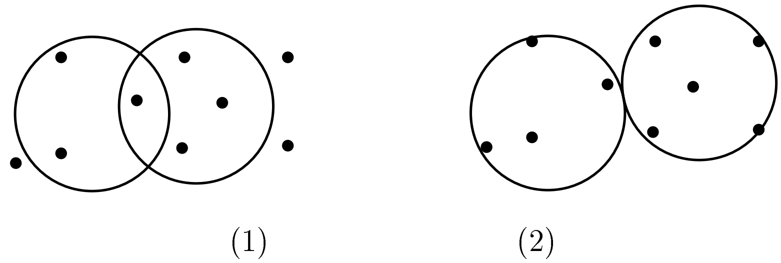


Fig. 6. The Process of Covering

IV. CONCLUSION

Two methods for Packing and covering problems are introduced in the paper. The shifting method, is very useful since it gives out the exact estimation of the errors and the speed, althogh the speed is sometimes only of theoratial interesting, since the degree of the polynomial is so high that it is also very slow normally (only be regarded fast compared to potential time complexity). The quasiphysical method, on the other hand can not give a exact estimation of the errors as well as the speed, but since it has a very hard phishycal and natural background, it always of useful in practice.

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