Trisection angles

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Abstract

With the new tools or method that beyond the classical Greek sense, it is possible to construct some problem that classical Greek construction method can not.

As an example, this article introduces several methods that can be employed to trisection angles. We also introduce a method to squareing a circle.

Keywords

Gemometry, Trisection

I. Introduction

Angle trisection, the division of an arbitary angle into three equal angles, is one of the three geometric problems of antiquity for which solusions by classical Greek construction method, that is using only Campass and Stright edge, were sought. It is proved firstly imposible by Wantzel in 1836 [6].

In this paper, however, we introduce several curves other then comapass and strightedges to accomplish the tasks, as well as a construcion operation that can also be employed to do so. We also show that the curves can be regard as constructable tools as compass and strightedges, by present a mechanisim for drawing Archimides spiral, the most famous one among the curves.

II. Marked Ruler

In classical Greek sence, the campass can only be used to draw a circle with a given point as center and a give segement as the radius, and the strightedge can only be used to draw lines pass through a pair of given points.

Mathmaticians have created a new construction, called “neusis” construction, or ”verging”, where the strightedge is a strightline being marked at two points on it and it is allowed to fit into a diagram so that the two points fall on two lines, perhaps curved. Neusis construction can be used to trisection angles as well as Cube Duplication, and construction of the rerular 7- 9- and 13-gons [2], [1]. In “Book of Lemmas”, Abu'l-Hasan Thabit ibn Qurra (826-901) had use the the neusis construction to trisection angles.

With the Marked Rule where the distance of the two marked points is r. Suppose the angle ABC is to be trisected. Draw a circle DEF with center B and radius r. Extend CB through D. Fit in a line DEF passing through D and a point F on the line BA so that the two points marked on the rule starts at a point E on the circle and ends at a point F in the line BA. (You'll have to move F around until E lands on the circle.)

By observe that $|BE| = |EF| = |BD|$, it is easy to know that the angle DFB is one third of ABC. (Figure 1)

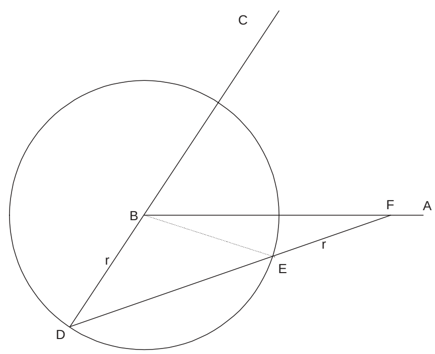


Fig. 1. Marked Ruler

III. Curves

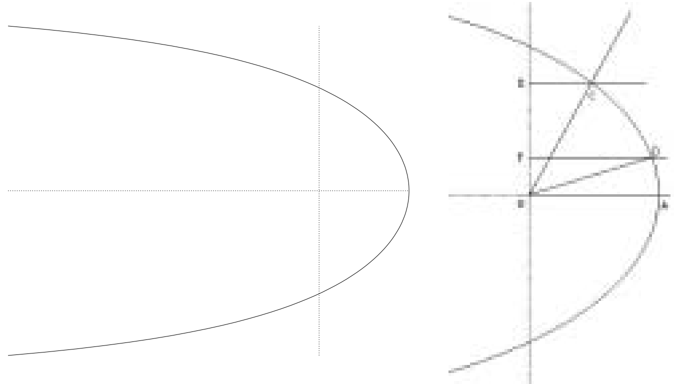
It is also posible to trisection angles by the help of other tools except the neusis construction. Some curves that can some how project arbitary angles into segements are the posible choices, since we have method to divide segement into any equal parts.

A. Quadratrix of Hippias

Quadratrix is conceived by Hippias of Elias (460 BC). The function of the curve could be described in polar system (figure 2(1)):

$$\rho = \theta / \sin(\theta)$$

To thisect the angle ABC , we first find the distance BE, then trisect the segment BE to get BC, then draw a horizental line intersect the curve at D. Since $|BE| = \sin(\angle ABC) \times$ |BC| $= \angle ABC/|BC| \times |BC| = \angle ABC$ and $|BF| = \angle ABD$, it is obvious that $\angle ABD$ is the trisection of $\angle ABC$. (figure 2(2))



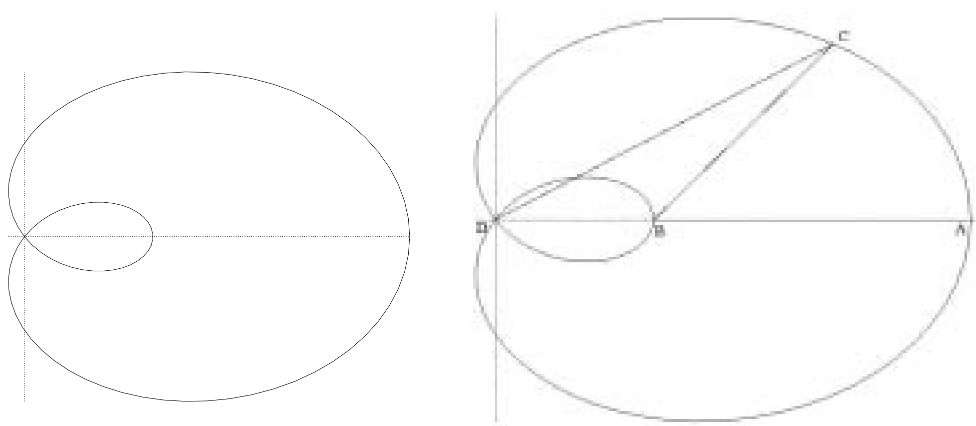
(1) (2)

Fig. 2. Quadratrix

B. Limacon of Pascal

Limacon of Pascal is a set of limacon (The name 'limacon' comes from the Latin limax meaning 'a snail') with expresion in polar as: $\rho = k + 2r\cos\theta$. It is discover by É. Pascal(1588-1651). A special case ot the limacon, which is $\rho = 1 + 2\cos\theta$ can be used to trisection(figure 3(1)).

Notice that $\tan(\angle BCD) = \sin(\angle BDC)/(1 + \cos(\angle BDC))$, so $\angle BCD = \frac{1}{2}\angle BDC$, the conclusion comes easily that $\angle BCD = \frac{1}{3}\angle ABC$ (figure 3(2)).



(1) (2)

Fig. 3. Limacon of Pascal

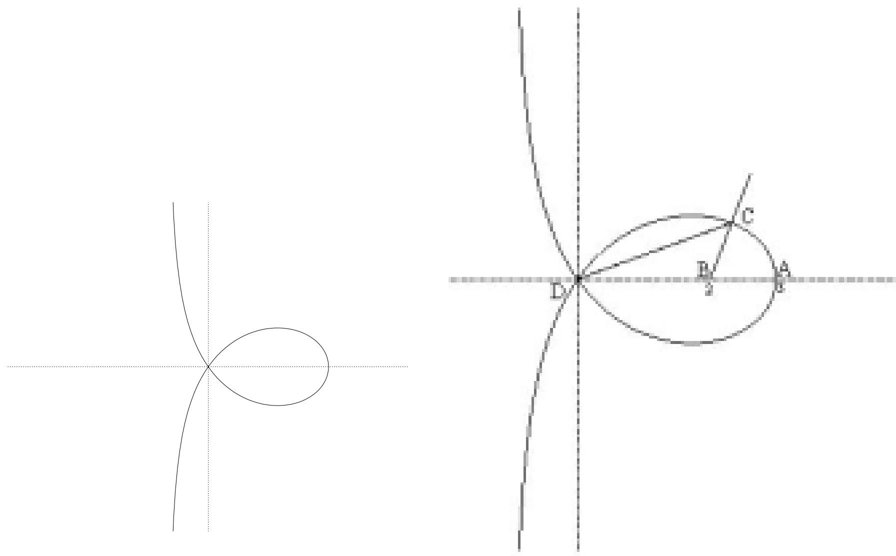
C. Trisectrix of Maclauri

This curve is used by colin Naclaurin (1698-1746). The expression in Polar system is (figture 4(1)):

$$\rho = \frac{2\sin 3\theta}{\sin 2\theta}$$

.

Notice that $\sin(\angle ABC)/\sin(\angle BCD) = |DC|/|DB| = \sin(3\angle ABC)/\sin(2\angle ABC)$, it is easy to conclude that $\angle ABC = 3\angle ADC$ (figture 4(2)).



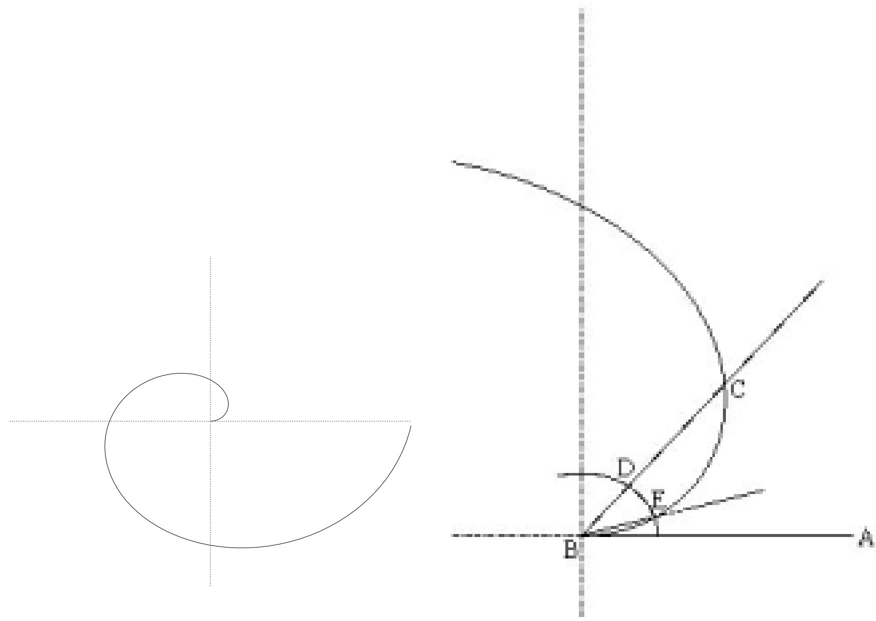
(1) (2)

Fig. 4. Trisectrix

D. Archimides Spiral

Archimedes proposed his spiral, in his "On Spirals" for the trisecting angles. The Spiril can be expressed as (figure 5(1)): $\rho = a\theta$.

Suppose the angle ABC is to be trisected. Trisect the segment BC and find —BD— be one third of —BC—, draw a circle with center B and rasius —BC—, suppose —BC— intersect the spiral at point E. Then it is easy to know that $\angle ABE = \frac{1}{3}\angle ABC$ (figure 5(2)).



(1) (2)

Fig. 5. Spiral

IV. The Mechanism for Archimides Spiral

It is obvious that the Marked rule, or we say "neusis" construction, is a good method to trisection angles. It is quite obvious that the marked ruler itself can be constructed as easily as the streightedge or compass. However, there is no golobal method can be employed to divide an angle into parts other then 3, by the marked ruler.

The above curves are useful to trisection any angles, among which the Quadratrix of Hoppias, the Archimides Spiral can also be used to divide the angles into any equal parts.Than the question comes, how to draw the curves then? It seems that we leave the problem of trisection angles to the problem of drawing the curves which may very difficult.

Here, we present an easy mechanism for draw the Archimide spiral (figure 6).

V. Further Application

With the ideas of new construction method or tools, we can also costruct another famous problem, squareing arbitary circles, that is draw a square that have the same area of a

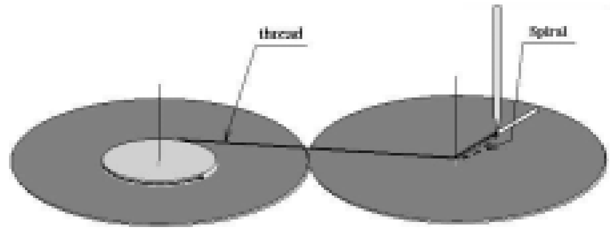


Fig. 6. Mechanism

given circle.

Theorem 1: If a square and a circle that have same area is given, then for any circles, we can square them in classic Greek sense.

This theorem can be easily proved, since when we know the two segments r_0 and a_0 satisfy $\pi r_0^2 = a_0^2$, then for any segement r, it is easy to construct another segment a such that $r/r_0 = a/a_0$. In this case we know $a^2 = r^2$.

Theorem 2: If we have been given two segment s_1 and s_2 such that $s_1/s_2 = \pi$, then the problem of squareing circles can be solved by classic Greek construction.

It is obvious since for any circle with a redius r, we can then construct a segment with lenth πr in classic sense, we can then construct a segment a such that $a^2 = \pi r \times r$.

Since in both the quadratrix and spiral can find the pair of such segments, we can sure use the these curves for squaring problem. Our mechanism has the same property.

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